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Total energy decay for the wave equation in exterior domains with a dissipation near infinity

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Abstract

We show that the energy of solutions to the initial boundary value problem for the wave equation in exterior domains with a dissipation which is localized only near infinity tends to zero as the time goes to infinity. We do not make any geometrical condition like star-shapedness on the boundary.

Keywords: Energy decay; Wave equations; Exterior domains; Dissipation

1. Introduction

Let us consider the exterior problem for the wave equation

$$u_{tt} - \Delta u + a(x)u_t = 0 \quad \text{in } \Omega \times [0, \infty), \tag{1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{and} \quad u|_{\partial\Omega} = 0, \tag{2}$$

where Ω is an exterior domain in $\mathbf{R}^N$ with a compact obstacle $V = \mathbf{R}^N/\Omega$. For $a(x)$ we assume:

Hypothesis A. $a(x)$ is a bounded nonnegative function on Ω and satisfies

$$a(x) \geqslant \epsilon_0 > 0 \quad \text{for } |x| \geqslant L,$$

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with some constants $\epsilon > 0$ and $L \gg 1$. (We may assume $V \subset B_L$.)

For each $(u_0, u_1) \in H_1^0(\Omega) \times L^2(\Omega)$ the problem admits a unique solution $u(t) \in C([0,\infty); H_1^0(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$, satisfying

$$\frac{d}{dt}E(t) + \int_\Omega a(x)\left|u_t(t)\right|^2 dx = 0, \tag{3}$$

where

$$E(t) = \frac{1}{2}\int_\Omega \left(\left|u_t(t)\right|^2 + \left|\nabla u(t)\right|^2\right) dx.$$

Thus $E(t)$ is decreasing as $t \to \infty$.

When V is star-shaped we know the uniform decay estimate

$$E(t) \leqslant C I_0^2 (1+t)^{-1}, \tag{4}$$

where $I_0^2 = \|u_0\|_{H_1}^2 + \|u_1\|^2$ (cf. Mochizuki and Nakazawa [6]). Note that estimate (4) never implies an exponential decay of $E(t)$ since I_0^2 includes $\|u_0\|^2$, L^2 norm of u_0.

More generally, following Russell [11] we set for $x_0 \in R^N$,

$$\Gamma(x_0) = \left\{x \in \partial\Omega \mid (x - x_0)\cdot \nu(x) > 0\right\},$$

where $\nu(x)$ denotes the outward unit normal vector w.r.t. Ω at $x \in \partial\Omega$, and assume in addition to Hypothesis A that there exist x_0 and ω, an open set in $\bar{\Omega}$, such that

$$\Gamma(x_0) \subset \omega \quad \text{and} \quad a(x) \geqslant \epsilon_0 > 0 \quad \text{for } x \in \omega. \tag{5}$$

Then we can prove the same estimate (4) (cf. Nakao [7]). Here we note that V is star-shaped with respect to x_0 if and only if $\Gamma(x_0) = \emptyset$.

Concerning the decay of local energy $E_{\mathrm{loc}}(t)$ for the usual wave equation (i.e., Eq. (1) without $a(x)u_t$) it is well known that $E_{\mathrm{loc}}(t)$ tends to 0 as $t \to \infty$ in general exterior domains (cf. Mizohata [4], Iwasaki [2], Morawetz et al. [5]). Therefore it seems natural to ask whether or not the total energy $E(t)$ tends to 0 as $t \to \infty$ under Hypothesis A, without assumption (5). The object of this note is to give an affirmative answer to this question.

When V is trapping we cannot expect any uniform decay of $E_{\mathrm{loc}}(t)$ (Ralston [10]) and hence it seems difficult to expect the uniform decay of $E(t)$ under Hypothesis A only.

For generality we consider in fact the wave equation with a nonlinear dissipation

$$u_{tt} - \Delta u + \rho(x, u_t) = 0 \quad \text{in } \Omega \times [0,\infty), \tag{1$'$}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{and} \quad u|_{\partial\Omega} = 0, \tag{2$'$}$$

where we make the following assumption on $\rho(x, v)$.

Hypothesis $\tilde{\mathbf{A}}$. $\rho(x, v)$ is a differentiable nondecreasing function in $v \in \mathbf{R}$ for a.e. $x \in \Omega$ and satisfies the conditions:

$$\rho(x,0) = 0, \qquad \rho(x,v) = a(x)v \quad \text{if } |x| \geqslant L$$

and

$$k_0 a(x)|v|^{r+2} \leqslant \rho(x,v)v \leqslant k_1 a(x)\left(|v|^2 + |v|^{r+2}\right) \quad \text{for } x \in \Omega \cap B_L,\ v \in \mathbf{R},$$

with $k_0, k_1 > 0$, $0 \leqslant r \leqslant 2/(N-2)^+$ $(0 \leqslant r < \infty$ if $N = 1, 2)$, where $a(x)$ is a nonnegative function satisfying Hypothesis A.

Under Hypothesis $\tilde{\mathrm{A}}$ and the additional assumption (5) we can prove for general domains that the solution $u(t)$ of (1′)–(2′) with $(u_0, u_1) \in H^2 \cap H_0^1 \times H_0^1$ satisfies

$$E(t) \leqslant C\big(\|u_0\|_{H_2} + \|u_1\|_{H_1}\big)(1+t)^{-1+\alpha}$$

with a certain $0 < \alpha < 1$. (See Nakao and Jung [9].) This estimate is more delicate than (4). For convenience we call the dissipation $\rho(x, u_t)$ satisfying Hypothesis $\tilde{\mathrm{A}}$ as 'half-linear' dissipation. Here, under Hypothesis $\tilde{\mathrm{A}}$ only we shall prove that $\lim_{t\to\infty} E(t) = 0$ for the solutions of problem (1′)–(2′). Our result reads as follows:

Theorem 1. *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then, under Hypothesis $\tilde{\mathrm{A}}$, there exists a unique solution $u(\cdot) \in X_1 = C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ with $\int_0^\infty \int_\Omega \rho(x, u_t)\, u_t \, dx\, ds < \infty$ of problem* (1′)–(2′) *and we have*

$$\lim_{t\to\infty} E(t) = 0.$$

2. Local energy decay

For any $R \gg 1$ we set $B_R \cap \Omega = \Omega_R$ and

$$E_R(t) = \frac{1}{2} \int\limits_{\Omega_R} \big(\big|u_t(t)\big|^2 + \big|\nabla u(t)\big|^2\big)\, dx.$$

In this section we prove:

Proposition 1. *For a solution $u(t)$ in Theorem* 1 *we have* $\lim_{t\to\infty} E_R(t) = 0$.

Proof. For the proof we use an argument similar to those for the equation in the bounded domains (cf. Dafermos [1], Iwasaki [2], Haraux [3]).

Let $(u_{0n}, u_{1n}) \in C_0^\infty(\Omega) \times C_0^\infty$ be a sequence of initial data such that

$$u_{0n} \to u_0 \quad \text{in } H_1^0 \quad \text{and} \quad u_{1n} \to u_1 \quad \text{in } L^2(\Omega) \quad \text{as } n \to \infty.$$

For (u_{0n}, u_{1n}) there exists a unique solution $u_n(t)$ of problem (1′)–(2′) in the class $X_2 = W^{2,\infty}_{\mathrm{loc}}([0,\infty); L^2(\Omega)) \cap W^{1,\infty}_{\mathrm{loc}}([0,\infty); H_0^1(\Omega)) \cap L^\infty_{\mathrm{loc}}([0,\infty); H^2 \cap H_0^1(\Omega))$. Further, by the finite propagation property we see that $u_n(t)$ has a compact support in $\mathbf{R}^N$ for each t. These are standard results.

Now, by the monotonicity of $\rho(x, v)$ in v we see from the equations that

$$\big\|u_{mt}(t) - u_{nt}(t)\big\|^2 + \big\|\nabla u_m(t) - \nabla u_n(t)\big\|^2 \leqslant \|u_{1m} - u_{1n}\|^2 + \|\nabla u_{0m} - \nabla u_{0n}\|^2,$$
$$t \geqslant 0,$$

and hence, $u_n(t)$ converges to a function $u(\cdot) \in C([0,\infty); H_1^0(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$. Note that the convergences of $u_{nt}(t)$ and $\nabla u_n(t)$ are uniform in $L^2(\Omega)$, and the convergence of $u_n(t)$ is uniform in $L^2(\Omega)$ on each interval $[0, T]$, $T > 0$. We see further

$$\rho(x, u_{nt}) \to \rho(x, u_t) \quad \text{weakly in } L^{(r+2)/(r+1)}\big([0,\infty); L^{(r+2)/(r+1)}(\Omega)\big).$$

Thus, $u(t)$ is the desired solution of $(1')$–$(2')$. (The lower bound of $\rho(x,v)v$ in Hypothesis $\tilde{\mathrm{A}}$ is used only here.)

Therefore, by a standard density argument it suffices to show our assertion for the case $u(\cdot)\in X_2$.

We observe that

$$E(t)\leqslant E(0)<\infty \quad \text{for } t\geqslant 0, \tag{6}$$

and

$$\|u_{tt}(t)\|^2+\|\nabla u_t(t)\|^2+\|\Delta u(t)\|^2\leqslant C\big(\|u_0\|_{H_2}^2+\|u_1\|_{H_1}^2\big)<\infty, \quad 0\leqslant t<\infty. \tag{7}$$

Further, by Poincaré lemma,

$$\int\limits_{\Omega_R}|u(t)|^2\,dx\leqslant C(R)\|\nabla u(t)\|^2\leqslant C<\infty.$$

Therefore, by Rellich's theorem, for any sequence $\{t_n\}\to\infty$ as $n\to\infty$ there exists a subsequence such that

$$u(t+t_n)\to\tilde{u}(t)\quad \text{strongly in } C_{\mathrm{loc}}\big(R;H^1_{0,\mathrm{loc}}(\Omega)\big),$$
$$u_t(t+t_n)\to\tilde{u}_t(t)\quad \text{strongly in } C_{\mathrm{loc}}\big(R;L^2_{\mathrm{loc}}(\Omega)\big)$$

and

$$\rho\big(x,u_t(t+t_n)\big)\to\rho\big(x,\tilde{u}_t(t)\big)\quad \text{weakly in } L^{(r+2)/(r+1)}_{\mathrm{loc}}\big(\mathbf{R};L^{(r+2)/(r+1)}(\Omega)\big).$$

We know

$$\tilde{u}\in C\big(\mathbf{R};H^2_{\mathrm{loc}}(\Omega)\cap H^1_{0,\mathrm{loc}}(\Omega)\big)\cap C^1_{\mathrm{loc}}\big(\mathbf{R};H^1_{0,\mathrm{loc}}(\Omega)\big)\cap C_{\mathrm{loc}}\big(\mathbf{R};L^2_{\mathrm{loc}}(\Omega)\big)$$

and

$$\tilde{u}_{tt}-\Delta\tilde{u}(t)+\rho(x,\tilde{u}_t)=0\quad \text{in } \Omega\times\mathbf{R}. \tag{8}$$

But, for any $T>0$ fixed, we have for large n

$$E(T+t_n)+\int\limits_{t_n-T}^{t_n+T}\int\limits_{\Omega}\rho(x,u_t)u_t(t)\,dx\,dt=E(-T+t_n).$$

Since $\lim_{n\to\infty}E(T+t_n)=\lim_{n\to\infty}E(-T+t_n)$ and $u_t(t+t_n)\to\tilde{u}_t(t)$ strongly in $L^2_{\mathrm{loc}}([-T,T]\times\Omega)$ we know by Fatou's lemma that

$$\int\limits_{-T}^{T}\int\limits_{\Omega}\rho\big(x,\tilde{u}_t(t)\big)\tilde{u}_t(t)\,dx\,dt=0.$$

Since T is arbitrary we conclude that

$$\tilde{u}_{tt}(t)-\Delta\tilde{u}(t)=0\quad \text{in } \Omega\times\mathbf{R} \tag{9}$$

and

$$\tilde{u}_t=0\quad \text{on } \operatorname{supp}a(\cdot)\times\mathbf{R}.$$

Setting $\tilde{u}_t(t) = v$ (note that all of the first and second derivatives of $\tilde{u}$ are bounded in $L^2(\Omega)$ we see $v \in C(\mathbf{R}; H_0^1) \cap C^1(\mathbf{R}; L^2)$) and

$$v_{tt} - \Delta v = 0 \quad \text{in } \Omega_R \times \mathbf{R} \tag{10}$$

with $v|_{\partial\Omega_R} = 0$ for all $R > L$.

Since $v(t)$ is an $H_0^1(\Omega_R)$ valued almost-periodic function on $\mathbf{R}$ satisfying

$$v(x,t) = 0 \quad \text{on } \operatorname{supp} a(\cdot) \times \mathbf{R},$$

we conclude that $v(x,t) \equiv 0$ on $\Omega \times \mathbf{R}$ (Dafermos [1], Iwasaki [2]). Therefore, we see $\tilde{u}(x,t) = \tilde{u}(x)$, independent of t, and

$$-\Delta\tilde{u}(x) = 0, \quad \tilde{u} \in \dot{H}_0^1(\Omega), \tag{11}$$

where $\dot{H}_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla u\|$. This implies that $\|\nabla\tilde{u}(x)\| = 0$ and consequently $\tilde{u}(x) \equiv 0$.

Thus, we conclude that $E_R(t)$ is convergent to 0 as $t \to \infty$. □

Remark 1. For the proof of Proposition 1 the 'half-linearity' is not used, and the result can be applied under a weaker situation, for example, if $\rho(x,v)$ satisfies the latter condition in Hypothesis Ã on Ω with $a(x)$ such that $a(x) > 0$ for $|x| \geqslant L$.

Remark 2. If we assume $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ the solution $u(t)$ in X_2 exists under the weaker assumption

$$\rho(x,0) = 0, \qquad 0 \leqslant \rho_v(x,v) \quad \text{and} \quad \rho(x,v)v \leqslant k_1\big(|v|^2 + |v|^{r+2}\big)$$

with $0 \leqslant r \leqslant 2/(N-2)^+$ and the conclusion of Proposition 1 holds for this solution under the additional condition $\rho(x,v)v > 0$ for $|x| \geqslant L$, $v \neq 0$.

3. Proof of theorem

Let $\phi(x)$ be a smooth function satisfying

$$\phi(x) = 1 \quad \text{for } |x| \geqslant 2L \quad \text{and} \quad \phi(x) = 0 \quad \text{for } x \in \Omega_L.$$

Setting $w = \phi(x)u$ we have

$$w_{tt} - \Delta w + a(x)w_t = -\nabla\phi \cdot \nabla u - \Delta\phi u \equiv f(x,t), \tag{12}$$

where we have used the half-linearity of $\rho(x,v)$.

We set

$$E_w(t) = \frac{1}{2}\big(\|w_t(t)\|^2 + \|\nabla w(t)\|^2\big).$$

Proposition 2. *There exists $\{t_n\}$ such that $t_n \to \infty$ and $E_w(t_n) \to 0$ as $n \to \infty$.*

Proof. Since $a(x) \geqslant \epsilon_0 > 0$ on B_L^c we have from (12) that

$$\frac{d}{dt}E_w(t) + \frac{\epsilon_0}{2}\int_{B_L^c} |w_t|^2\,dx \leqslant \frac{1}{2}\int_{B_{2L}} \big|f(x,t)\big|^2\,dx \tag{13}$$

and also, by multiplying the equation by w,

$$\frac{d}{dt}\left\{(w_t, w) + \frac{1}{2}\int_{B_L^c} a(x)|w|^2\,dx\right\} + \|\nabla w\|^2 = (w, f) + \int_{B_L^c} |w_t|^2\,dx. \tag{14}$$

Note that

$$\int_{B_{2L}} |w|^2 dx \leqslant C\|\nabla w\|^2 \tag{15}$$

with some $C > 0$.

Then combining (13) and (14), we have

$$\frac{d}{dt}X(t) + E_w(t) \leqslant C\left(\|f(t)\|^2 + \int_{B_L} |w_t|^2\,dx\right), \tag{16}$$

where

$$X(t) = kE_w(t) + (w_t, w) + \frac{1}{2}\int_{B_L^c} |w|^2\,dx$$

with $k > 4\epsilon_0^{-1}$.

We again use (15) to observe that $X(t)$ is equivalent to $E_w(t) + \|w(t)\|^2$ for a large $k > 0$.

Now, if our assertion is not true there would exist $\delta > 0$ and $T_0 > 0$ such that $E_w(t) \geqslant \delta$ for $t > T_0$. By Proposition 1 we know that

$$\|f(t)\|^2 \leqslant C\int_{\Omega_{2L}} \left(|u(t)|^2 + |\nabla u(t)|^2\right)dx \leqslant C\int_{\Omega_{2L}} |\nabla u(t)|^2\,dx \to 0 \quad \text{as } t \to \infty$$

and

$$\int_{B_L} |w_t(t)|^2\,dx \leqslant C\int_{B_L} |u_t(t)|^2\,dx \to 0 \quad \text{as } t \to \infty.$$

Hence, there exists $T_1 \geqslant T_0$ such that

$$\frac{d}{dt}X(t) + \frac{1}{2}E_w(t) \leqslant 0, \quad t > T_1, \tag{17}$$

which implies

$$\int_{T_1}^{\infty} E_w(t)\,dt \leqslant 2X(T_1) \leqslant C\left(T_1, \|u_0\|_{H_1}, \|u_1\|\right) < \infty.$$

This is a contradiction to the claim that $E_w(t) \geqslant \delta > 0$, $t > T_0$. □

Completion of the proof of theorem. Using $\phi(x)$ in the above we have

$$\begin{aligned} E(t) &= \frac{1}{2}\left\{\left\|(1-\phi)u_t(t) + \phi u_t(t)\right\|^2 + \left\|(1-\phi)\nabla u(t) + \phi\nabla u(t)\right\|^2\right\} \\ &\leqslant CE_R(t) + 2E_w(t), \end{aligned}$$

where $R = 2L$.

Combining this with Propositions 1, 2, we know that $E(t_n) \to 0$ as $n \to \infty$ for some $\{t_n\}$. Since $E(t)$ is monotone decreasing we conclude that $E(t) \to 0$ as $t \to \infty$. $\square$

Remark 3. Our theorem is true also for the Klein–Gordon type wave equation

$$u_{tt} - \Delta u + m_0 u + \rho(x, u_t) = 0$$

with $m_0 > 0$. Indeed, in the proof of Theorem 1, we have only to replace $E(t)$ by $E(t) = \frac{1}{2}\int_\Omega (|u_t(t)|^2 + |\nabla u(t)|^2 + m_0|u(t)|^2)\,dx$ and make some trivial modifications. Note that any solution $u(t) \in C(\mathbf{R}; H_1^0(\Omega_R)) \cap C^1(\mathbf{R}; L^2(\Omega_R))$ of the problem $u_{tt} - \Delta u + m_0 u = 0$ in $\mathbf{R} \times \Omega_R$ is also an $H_1^0(\Omega_R)$ valued almost-periodic function. For the decay rates of energy of the Klein–Gordon equation with a localized dissipation in the whole space R^N, see Zuazua [12], Nakao [8].

References

[1] C. Dafermos, Asymptotic behaviour of solutions of evolution equations, in: M.G. Crandal (Ed.), Nonlinear Evolution Equations, Academic Press, New York, 1978, pp. 103–123.

[2] N. Iwasaki, Local decay of solutions for symmetric hyperbolic systems with dissipative and coercive boundary conditions in exterior domains, Publ. Res. Inst. Math. Sci. Kyoto Univ. 5 (1969) 193–218.

[3] A. Haraux, Stabilization of trajectories for some weakly damped hyperbolic equations, J. Differential Equations 59 (1985) 145–154.

[4] S. Mizohata, Theory of Partial Differential Equations, Cambridge Univ. Press, Cambridge, 1973.

[5] C. Morawetz, J. Ralston, W. Strauss, Decay of solutions of the wave equation outside non-trapping obstacles, Comm. Pure Appl. Math. 30 (1977) 447–508.

[6] K. Mochizuki, H. Nakazawa, Energy decay of solutions to the wave equations with linear dissipation localized near infinity, Publ. Res. Inst. Math. Sci. Kyoto Univ. 37 (2001) 441–458.

[7] M. Nakao, Energy decay for the linear and semilinear wave equations in exterior domains with some localized dissipations, Math. Z. 238 (2001) 781–797.

[8] M. Nakao, Decay of solutions to the Cauchy problem for the Klein–Gordon equation with a localized nonlinear dissipation, Hokkaido Math. J. 27 (1998) 245–271.

[9] M. Nakao, Il Hyo Jung, Energy decay for the wave equation in exterior domains with some half-linear dissipation, Differential Integral Equations 16 (2003) 927–948.

[10] J. Ralston, Solution of the wave equation with localized energy, Comm. Pure Appl. Math. 22 (1969) 807–823.

[11] D.L. Russell, Exact boundary value controllability theorems for wave and heat process in star-complimented regions, in: Roxin, Liu, Sternberg (Eds.), Differential Games and Control Theory, Marcel Dekker, New York, 1974.

[12] E. Zuazua, Exponential decay for the semilinear wave equation with localized damping in unbounded domains, J. Math. Pures Appl. 70 (1992) 513–529.